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Continuous and discrete frames on Julia sets

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Abstract

A frame is an overcomplete family of vectors in a Hilbert space in which the orthogonality condition is relaxed. The Julia set is the chaotic regime of a rational function. In this note, we label frames of an abstract Hilbert space by elements of the Julia set of a rational function.

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1. Introduction

Hilbert spaces are the natural frame work for the mathematical version of many areas of physics, predominantly quantum mechanics and signal and image analysis. Often the main problem is to decompose an arbitrary vector in terms of simpler vectors. What are these simpler vectors and how efficient are they? The natural choice for these vectors is an orthonormal basis of the Hilbert space of the problem. In practice, orthonormal bases are difficult to work with because they decompose the vector in a unique way. This lack of flexibility was not welcomed by practitioners in the field. By giving up the orthogonality of the basis vectors and thereby the uniqueness of the decomposition a more flexible alternative was introduced in 1952 by Duffin and Schaeffer [8] in the context of non-harmonic Fourier analysis. This alternative is called a frame.

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Iterations of rational functions on the Riemann sphere is another rapidly growing subject. It was first studied by Fatou and Julia when they independently discovered the dichotomy of the Riemann sphere into sets now bearing their names. For more information about holomorphic dynamics, we refer the reader to Refs. [3,7,9]. Interesting links between holomorphic dynamics and other mathematical areas have been found, for instance, Kleinian groups and number theory [7]. This motivates us to find a link between the two important mathematical topics: frames and holomorphic dynamics.

In this note, we label frames of an abstract Hilbert space by elements of the Julia set of a rational function. The paper is organized as follows. In Section 2, we present some definitions from holomorphic dynamics. In Section 3, we recall the definition of a frame. Then, we prove the main result of this note and present examples of rational functions which satisfy our conditions. In Section 4, we obtain a resolution of the identity from the frames which are labeled by the elements of a Julia set. A discrete frame on Julia set is obtained in Section 5. In Section 6 we present some results on Hilbert spaces using the properties of Julia sets. In Section 7, we discuss possible applications of our problem to brain imaging.

2. Holomorphic dynamics and the Julia set

In this section, we state some definitions and well known results which will be used in the sequel. All the results can be found in the above references.

Definition 2.1. A family of analytic functions having a common domain of definition is called normal if every sequence in this family contains a locally uniformly convergent subsequence.

Definition 2.2. Let $Q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree greater than 1. We denote the nth iterate of Q by Q^n, i.e.:

$$Q^n = \underbrace{Q \circ Q \cdots \circ Q}_{n \text{ times}}.$$

The Fatou set is defined by

$$\mathcal{F}(Q) = \{z : \exists \text{ a neighborhood } U_z \text{ s.t. } \{Q^n\}_{n=1}^{\infty} \text{ is a normal family in } U_z\}.$$

The Julia set is defined by $J = J(Q) = \hat{\mathbb{C}} \backslash \mathcal{F}(Q)$.

The Julia set has the following properties:

(i) The Julia set is a compact set.
(ii) The Julia set is a perfect set.
(iii) The Julia set is completely invariant under Q, i.e., for any $z \in J(Q)$ we have $Q^n(z) \in J(Q)$, $n = 0, \pm 1, \pm 2, \ldots$

Recently, more attention has been called for the study of measures and dimensions of fractals; in particular fractals obtained from Julia sets [10]. One of the most famous measures defined on the Julia set is called conformal measure. We first state the definition of conformal measures defined on general compact measure spaces, then we specify it to Julia sets.

Definition 2.3. Let $T : X \to X$ be a continuous mapping of a compact metric space (X, ρ) and let $h : X \to \mathbb{R}$ be a non-negative measurable function. A Borel probability measure μ on X is said to be h-conformal for $T : X \to X$ if

$$\mu(T(A)) = \int_A h \, \mathrm{d}\mu$$

for any Borel set $A \subset X$ such that $T|_A$ is injective and $T(A)$ is measurable. Such sets are called special sets.

Let $Q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. Since the Julia set, $J(Q)$, is Q-invariant, we can define $Q : J(Q) \to J(Q)$.

Definition 2.4. Let $t \geq 0$. Any $|Q'|^t$-conformal measure for $Q : J \to J$ is called t-conformal measure, i.e.:

$$\mu(Q(A)) = \int_A |Q'|^t \, \mathrm{d}\mu$$

for every special set $A \subset J$.

In order to motivate the definition of the conformal measure notice that if $J(Q) = \hat{\mathbb{C}}$ and $t = 2$, then the normalized Lebesgue measure is 2-conformal. Even more, if the t-dimensional Hausdorff measure is finite and positive on the Julia set, then the corresponding measure is t-conformal [10]. In [11], Sullivan showed that for every rational function there exists a conformal measure. In [4,5], a general scheme of constructing conformal measures was found and applied to the case of Sullivan measures.

3. Continuous frames on Julia sets

Definition 3.1. Let (X, μ) be a locally compact measure space and $\mathfrak{H}$ be an abstract separable Hilbert space. The family of vectors

$$\mathfrak{S} = \{|\eta_x\rangle | x \in X\} \subset \mathfrak{H}$$

is said to form a frame in $\mathfrak{H}$ if the operator:

$$F = \int_X |\eta_x\rangle\langle\eta_x| \, \mathrm{d}\mu \tag{3.1}$$

satisfies

$$A\|\phi\|^2 \leq \langle\phi|F\phi\rangle \leq B\|\phi\|^2 \quad \text{for all } \phi \in \mathfrak{H}, \tag{3.2}$$

where A and B are positive constants. If $A = B$ the set $\mathfrak{S}$ is called a tight frame. If the operator $F = I$, the identity operator of $\mathfrak{H}$, then the set $\mathfrak{S}$ is said to give a resolution of the identity. Note that if $X = \Gamma$ is a discrete set and μ is a counting measure, the operator (3.1) takes the form

$$F = \sum_{j\in\Gamma} |\eta_j\rangle\langle\eta_j|. \tag{3.3}$$

In the case where X is partly discrete, the corresponding part of μ could in general be a weighted counting measure and (3.1) takes the form

$$F = \sum_{j\in\Gamma'} \int_{X'} |\eta_{x,j}\rangle\langle\eta_{x,j}|\,\mathrm{d}\nu(x), \tag{3.4}$$

where $X = X' \cup \Gamma'$, X' is the continuous part with measure ν and Γ' the discrete part with a counting measure on it.

Let $Q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and $J(Q)$ its corresponding Julia set. Let μ be a conformal measure defined on the Julia set:

$$0 < \int_J \mathrm{d}\mu(z) = M < \infty. \tag{3.5}$$

We assume

$$A \le |Q^m(z)|^2 \le B \quad \text{for all}\, m = 0, 1, 2, \ldots \text{ and for all}\, z \in J, \tag{3.6}$$

where A and B are positive constants.

Any separable Hilbert space, $\mathfrak{H}$, possesses an orthonormal basis. In this note, $\mathfrak{H}$ is an abstract separable Hilbert space and $\{\phi_m\}_{m=0}^{\infty}$ is an orthonormal basis of $\mathfrak{H}$. In the finite-dimensional Hilbert space case, an orthonormal basis is $\{\phi_m\}_{m=0}^{N}$, where N is the dimension of the Hilbert space.

Theorem 3.2. *For $z \in J$ let*

$$|\phi_{z,m}\rangle = Q^m(z)\phi_m \tag{3.7}$$

and $S = \{|\phi_{z,m}\rangle | z \in J, m = 0, 1, 2, \ldots\}$.

If condition (3.6) is satisfied, the set S constitute a frame in $\mathfrak{H}$. That is, the operator

$$F = \sum_{m=0}^{\infty} \int_J |\phi_{z,m}\rangle\langle\phi_{z,m}|\,\mathrm{d}\mu(z)$$

satisfies

$$K\|\phi\|^2 \le \langle\phi|F\phi\rangle \le L\|\phi\|^2$$

for all $\phi \in \mathfrak{H}$ and some positive constants K and L.

Proof. We have

$$\begin{aligned}\langle\phi|F\phi\rangle &= \sum_{m=0}^{\infty}\int_J \langle\phi|\phi_{z,m}\rangle\langle\phi_{z,m}|\phi\rangle\,\mathrm{d}\mu(z)\\ &= \sum_{m=0}^{\infty}\int_J \langle\phi|Q^m(z)\phi_m\rangle\langle Q^m(z)\phi_m|\phi\rangle\,\mathrm{d}\mu(z)\\ &= \sum_{m=0}^{\infty}\int_J |Q^m(z)|^2\langle\phi|\phi_m\rangle\langle\phi_m|\phi\rangle\,\mathrm{d}\mu(z). \end{aligned} \tag{3.8}$$

Thus

$$\langle\phi|F\phi\rangle = \sum_{m=0}^{\infty}\int_J |Q^m(z)|^2|\langle\phi|\phi_m\rangle|^2\,\mathrm{d}\mu(z). \tag{3.9}$$

By (3.5), (3.6) and (3.9) we obtain

$$\langle\phi|F\phi\rangle \leq MB^2\sum_{m=0}^{\infty}|\langle\phi|\phi_m\rangle|^2 = MB^2\|\phi\|^2 = L\|\phi\|^2$$

and

$$\langle\phi|F\phi\rangle \geq MA^2\sum_{m=0}^{\infty}|\langle\phi|\phi_m\rangle|^2 = MA^2\|\phi\|^2 = K\|\phi\|^2.$$

This ends the proof. □

In the setting of signal processing, $\mathfrak{H}$ is a Hilbert space of finite energy signals and $(J(Q), \mathrm{d}\mu)$ is a measure space of parameters. Every signal contains noise but the nature and the amount of noise is different for different signals. In this context, choosing $(J(Q), \mathrm{d}\mu, \{|\phi_{z,m}\rangle\})$ amounts to selecting a part of the signal that we wish to isolate and interpret, while eliminating a noise that has been deemed useless. In this process, what we have done, in effect, is chosen a frame on $\mathfrak{H}$ living in $J(Q)$ (for details see [1]). Further, in mathematical terms, a signal is a square integrable function. Thus one is allowed to choose $\mathfrak{H}$ to be $L^2(J(Q), \mathrm{d}\mu)$ and $\{\phi_m\}$ to be an orthonormal basis of it. Since the construction works on any separable Hilbert space, we do not articulate this point any further. In Theorem 3.2, we have used the Julia set of Q to build a flexible alternate, frame, for the orthonormal basis.

Remark 3.3. In the terminology of [2], the frames constructed in Theorem 3.2 can be named as *continuous frames of infinite rank*.

Now, we present examples of families of rational maps which satisfy assumption (3.5) and hence Theorem 3.2. These examples are chosen as models which are well studied in the literature [3,7,9]. From the above theorem, it is clear that any Julia set $J(Q)$ having the property $N_0 \cap J(Q) = \emptyset$, where N_0 is a neighborhood of zero, admits a frame. Moreover, the first example is chosen to give a *good* frame (this point will be clarified later in the text).

Example 3.4. Let $Q(z) = \mathrm{e}^{\mathrm{i}\alpha}\Pi_{j=1}^{k}(z-a_j)/(1-\bar{a}_j z)$, where $|a_j| < 1$ and a_j is a complex number for $j = 1, \ldots, k$. $Q(z)$ is known as Blaschke product. The following is well known fact about the Julia sets of Blaschke products [9]:

The Julia set of a Blaschke product is S^1 *if and only if there exists* $z_0 \in \mathbb{D}$ *such that* $Q(z_0) = z_0$, *where* $\mathbb{D}$ *is the unit disc and* S^1 *the unit circle.*

Thus, when $Q(z_0) = z_0$ for some $z_0 \in \mathbb{D}$, $Q(z)$ clearly satisfies assumption (3.6). For the conformal measure μ, we take Lebesgue arc measure. Observe that, since we are on the unit circle

$|Q^m(z)|^2 = 1$ and

$$\int_J \mathrm{d}\mu(z) = 2\pi$$

under the assumed measure. Thus

$$\langle\phi|F\phi\rangle = 2\pi\|\phi\|^2.$$

That is, we obtain a tight frame.

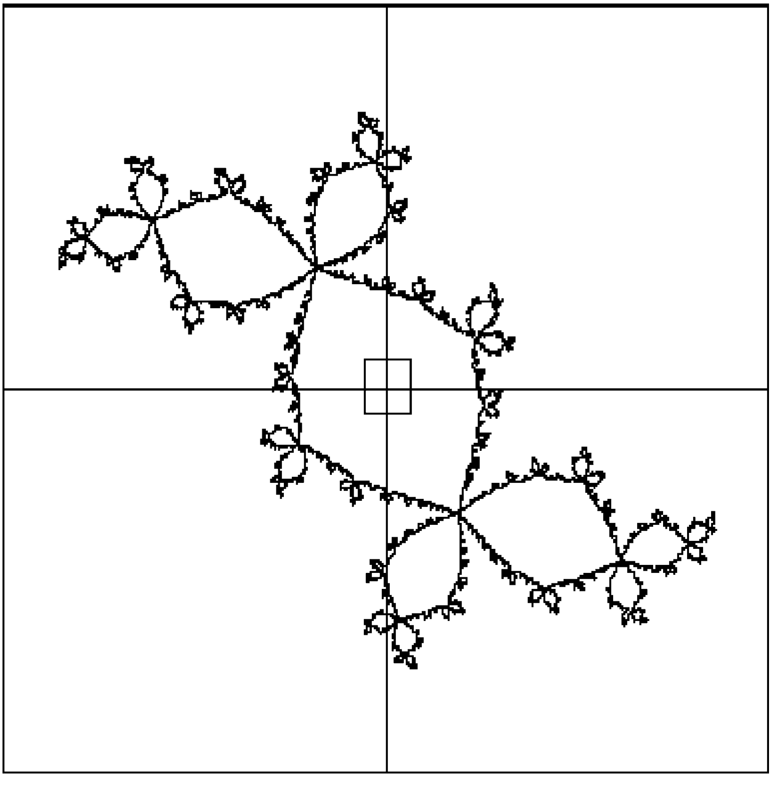

Fig. 1. $Q(z) = z^2 - 0.1 + 0.8\mathrm{i}$.

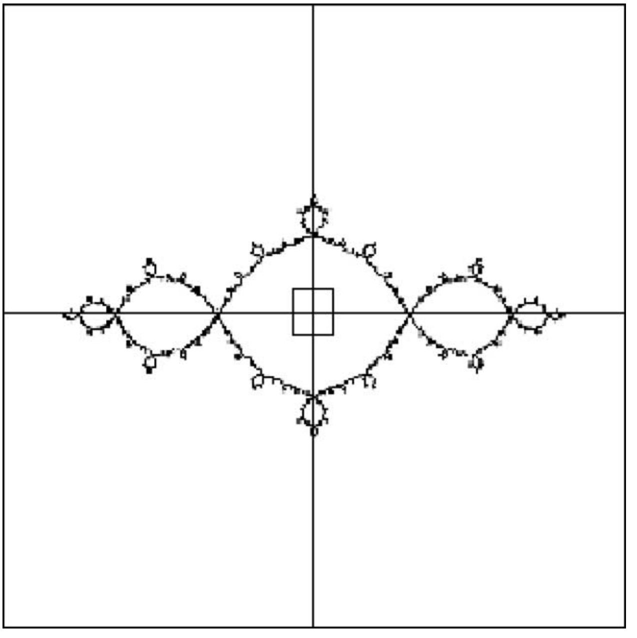

Fig. 2. $Q(z) = z^2 - 1$.

Example 3.5. The quadratic family, $Q(z) = z^2 + c$, is the most famous family in holomorphic dynamics. Although it is the simplest non-linear example, it has very rich and complicated dynamics. Its Julia set is always symmetric with respect to the origin and always contained within the circle $|z| = 2$. Replacing c by its conjugate has the effect of reflecting J_c through the horizontal axis [3,7,9]. Now, we give examples of Julia sets of the quadratic family that satisfy (3.6).

The simplest case is when $c = 0$, where the Julia set is the unit circle. For $c = -0.1+0.8\mathrm{i}$ the Julia set of $Q(z)$ is called the *Douady Rabbit*. The Julia set is symmetric with respect to the origin which does not belong to the Julia set. Hence, we can draw a neighborhood N_0 around the origin and inside the Julia set which is bounded inside the circle $|z| = 2$ (see Fig. 1).

Therefore, there are constants A and B such that (3.6) is satisfied. Thus, we have frames labeled by the iterates of $Q(z)$ in the sense of Theorem 3.2. Another, interesting example of the quadratic family is $Q(z) = z^2 - 1$ where the Julia set has the shape of towers. This case is similar to the Douady Rabbit case (see Fig. 2).

Now, we present an example of the quadratic family which does not satisfy condition (3.6).

Example 3.6. Let $Q(z) = z^2 - 2$. The Julia set of this function is the closed interval $[-2, 2]$. Here we cannot have a neighborhood N_0 of the origin such that $N_0 \cap J = \emptyset$. Thus, $Q(z) = z^2 - 2$ does not satisfy the assumption of Theorem 3.2.

4. Resolution of the identity

In this section we obtain a resolution of the identity from frames. This procedure is standard. We formulate it to our problem. From the frame condition (3.2), it is clear that the frame operator F, the resolution operator in the terminology of signal processing, is bounded and has a bounded inverse. F allows us to decompose any vector $\phi \in \mathfrak{H}$ as

$$F\phi = \sum_{m=0}^{\infty} \int_{J(Q)} \langle \phi_{m,z} | \phi \rangle \phi_{m,z} \, \mathrm{d}\mu(z).$$

However, in order to have a direct decomposition one prefers to have the operator F to be I, the identity operator on $\mathfrak{H}$.

Theorem 4.1. *The operator F of* Theorem 3.2 *is a bounded invertible operator with a bounded inverse. Furthermore, it is self-adjoint.*

Proof. The boundedness and the boundedness of the inverse follows from the frame condition. For the self-adjointness see [2]. □

Let us define a new class of states as follows:

$$|\eta_{z,m}\rangle = F^{-1/2} |\phi_{z,m}\rangle = |F^{-1/2} \phi_{z,m}\rangle. \tag{4.1}$$

Let

$$\Omega = \{ |\eta_{z,m}\rangle | z \in J, m = 0, 1, 2, \ldots \}.$$

Theorem 4.2. *The set Ω gives a resolution of the identity.*

Proof. Since F is self-adjoint, $F^{-1/2}$ is also self-adjoint. Consider

$$\begin{aligned}
&\sum_{m=0}^{\infty} \int_J |\eta_{z,m}\rangle\langle \eta_{z,m}| \, \mathrm{d}\mu(z) \\
&\quad = \sum_{m=0}^{\infty} \int_J |F^{-1/2}\phi_{z,m}\rangle\langle F^{-1/2}\phi_{z,m}| \, \mathrm{d}\mu(z) \\
&\quad = F^{-1/2} \sum_{m=0}^{\infty} \left(\int_J |\phi_{z,m}\rangle\langle \phi_{z,m}| \, \mathrm{d}\mu(z) \right) F^{-1/2} = F^{-1/2} F F^{-1/2} = I.
\end{aligned}$$

Thus, the collection Ω gives a resolution of the identity. □

Let us answer the following question: is there a sequence $\{f_m(z)\}_{m=0}^{\infty}$ such that

$$|\eta_{z,m}\rangle = f_m(z)\phi_m. \tag{4.2}$$

Since

$$|\eta_{z,m}\rangle = F^{-1/2} |\phi_{z,m}\rangle$$

we have

$$\langle\phi_m|\eta_{z,m}\rangle = \langle\phi_m|F^{-1/2}\phi_{z,m}\rangle.$$

From which we get

$$f_m(z) = \overline{Q^m(z)}\langle\phi_m|F^{-1/2}\phi_m\rangle. \tag{4.3}$$

In the cases where $F^{-1/2}$ is explicitly known we can reduce (4.3) to a closed form.

5. Discrete frames on Julia sets

The practical implementation of any continuous scientific process requires a discretization, all formulas must generally be evaluated numerically and a computer is intrinsically a discrete object, even finite. For example, in signal processing, the practical use of the wavelet transform requires the selection of a discrete set of points in the transform space. But this operation must be done in such a way that no information is lost. This requirement leads to the determination of a discrete frame [1]. In mathematical terms this reads: for some general space X given a continuous frame $\{\phi_x : x \in X\}$ can one find a discrete set of points $\{x_j : j \in L \subseteq \mathbb{N}\}$ such that $\{\phi_{x_j} : j \in L\}$ is a discrete frame, possibly with a different frame operator [1]. In the following we present it to our case.

Let $z_j \in J(Q)$ for $j = 0, 1, 2, \ldots, K < \infty$, then $Q^m(z_j) \in J$ for all $m = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots, K$. Consider the set

$$\mathfrak{S}_K = \{|\phi_{m,j}\rangle = Q^m(z_j)\phi_m : m = 0, 1, 2, \ldots, j = 0, 1, 2, \ldots, K\}.$$

We assume

$$A \leq |Q^m(z_j)|^2 \leq B, \tag{5.1}$$

where A, B are positive constants.

Theorem 5.1. *Suppose (5.1) is satisfied. Then, the set $\mathfrak{S}_K$ is a frame. That is, the operator*

$$F = \sum_{m=0}^{\infty}\sum_{j=0}^{K}|\phi_{m,j}\rangle\langle\phi_{m,j}|$$

satisfies

$$A^2K\|\phi\|^2 \leq \langle\phi|F\phi\rangle \leq B^2K\|\phi\|^2.$$

Proof. Consider

$$\langle\phi|F\phi\rangle = \sum_{m=0}^{\infty}\sum_{j=0}^{K}\langle\phi|\phi_{m,j}\rangle\langle\phi_{m,j}|\phi\rangle = \sum_{m=0}^{\infty}\sum_{j=0}^{K}|Q^m(z_j)|^2|\langle\phi|\phi_m\rangle|^2.$$

Thus by (5.1)

$$\langle\phi|F\phi\rangle \le B^2 \sum_{m=0}^{\infty}\sum_{j=0}^{K} |\langle\phi|\phi_m\rangle|^2 = B^2 K \sum_{m=0}^{\infty} |\langle\phi|\phi_m\rangle|^2 = B^2 K \|\phi\|^2.$$

Again by (5.1)

$$\langle\phi|F\phi\rangle \ge A^2 \sum_{m=0}^{\infty}\sum_{j=0}^{K} |\langle\phi|\phi_m\rangle|^2 = A^2 K \sum_{m=0}^{\infty} |\langle\phi|\phi_m\rangle|^2 = A^2 K \|\phi\|^2.$$

Hence, $\mathfrak{S}_K$ is a discrete frame. □

Example 5.2. In the examples of the previous section one could pick a finite number of points from the Julia set.

In the above case the decomposition of a vector $\phi \in \mathfrak{H}$ reads

$$F\phi = \sum_{m=0}^{\infty}\sum_{j=1}^{K} \langle\phi_{m,z}|\phi\rangle\phi_{m,z}. \tag{5.2}$$

Once we have a frame, another question arises: how *good* is the frame? Here by *good* one means the fastness of convergence of the series (5.2). For the general frame condition (3.2) the quantity

$$\omega = \frac{B-A}{B+A}$$

called the width or snugness of a frame, measures the fastness (in fact the tightness of the frame). If $\omega = 0$ then the frame is tight. Thus a good frame means $\omega \ll 1$ [1,2].

In our case

$$\omega = \frac{B^2K - A^2K}{B^2K + A^2K} = \frac{B^2 - A^2}{B^2 + A^2}$$

and

$$A \le |Q^m(z)| \le B.$$

In Example 3.4 it is possible to pick A and B very close. Thus ω can be made arbitrarily close to zero. In this case we have a good frame. The Julia set of $Q(z) = z^2 + c$, for a very small c, is a perturbed circle. The same is true in this case.

Theorem 5.3. *If J is the Julia set of a Blaschke product, then for each $K \in \mathbb{N}$ the set $\mathfrak{S}_K$ constitute an orthonormal basis for $\mathfrak{H}$.*

Proof. Suppose J is the Julia set of a Blaschke product, then $A = B = 1$, i.e., $\mathfrak{S}_K$ is a tight frame. Further

$$\||\phi_{m,j}\rangle\|^2 = \langle Q^m(z_j)\phi_m|Q^m(z_j)\phi_m\rangle = |Q^m(z_j)|^2\langle\phi_m|\phi_m\rangle = 1.$$

Therefore, $\||\phi_{m,j}\rangle\| = 1$, for all $m = 0, 1, \dots, j = 1, 2, \dots, K$.

Since $A = B = 1$ we have

$$\||\phi_{m,k}\rangle\|^2 = \sum_{j=1}^{K}\sum_{n=0}^{\infty} |\langle\phi_{n,j}|\phi_{m,k}\rangle|^2 = \||\phi_{m,k}\rangle\|^4 + \sum_{j\neq k}\sum_{n\neq m} |\langle\phi_{n,j}|\phi_{m,k}\rangle|^2.$$

Now by $\||\phi_{m,j}\rangle\| = 1$, we get

$$\sum_{j\neq k}\sum_{n\neq m} |\langle\phi_{n,j}|\phi_{m,k}\rangle|^2 = 0,$$

which implies

$$\langle\phi_{n,j}|\phi_{m,k}\rangle = 0 \quad \forall\, n \neq m,\ j \neq k.$$

Therefore, $\mathfrak{S}_K$ is an orthonormal set, further, since $\mathfrak{S}_K$ is a frame, it is total in $\mathfrak{H}$. □

6. Properties of Hilbert spaces and Julia sets

In this section, we investigate the possibility of obtaining some properties of a Hilbert space from a Julia set and vice-versa.

Proposition 6.1. *For each $N \in \mathbb{N}$, the set*

$$\mathfrak{S}_N = \left\{ \psi_z = \sum_{j=0}^{N} Q^j(z)\phi_j : z \in J \right\} \tag{6.1}$$

is a closed subset of $\mathfrak{H}$.

Proof. Let $\{\psi_{z_k}\}_{k=0}^{\infty}$ be a sequence in $\mathfrak{S}_N$ with a limit. Then

$$\psi_{z_k} = \sum_{j=0}^{N} Q^j(z_k)\phi_j.$$

Now

$$\lim_{k\to\infty} \psi_{z_k} = \lim_{k\to\infty} \sum_{j=0}^{N} Q^j(z_k)\phi_j = \sum_{j=0}^{N} Q^j(\lim_{k\to\infty} z_k)\phi_j.$$

Since $\{z_k\}_{k=0}^{\infty} \subset J$ and J is compact, there exist $z_0 \in J$ such that $\lim_{k\to\infty} z_k = z_0$. We get

$$\lim_{m\to\infty} \psi_{z_k} = \sum_{j=0}^{N} Q^j(z_0)\phi_j = \psi_{z_0} \in \mathfrak{S}_N.$$

Thus $\mathfrak{S}_N$ is closed. □

Remark 6.2. Since for any $z_1, z_2 \in J$ and $\alpha, \beta \in \mathbb{C}$ there may not be a z_3 such that

$$\alpha Q^j(z_1) + \beta Q^j(z_2) = Q^j(z_3).$$

Therefore $\mathfrak{S}_N$, in general, is not a subspace.

Corollary 6.3. *For any finite $n \in \mathbb{N}$, the set*

$$\mathfrak{S}_1 = \bigcup_{\tau=1}^{n} \mathfrak{S}_{N_\tau} = \left\{ \psi_{z,\tau} = \sum_{j=0}^{N_\tau} Q^j(z)\phi_j : z \in J, \tau = 1, 2, \ldots, n \right\}$$

is a closed subset of $\mathfrak{H}$.

Proposition 6.4. *For each $N, M \in \mathbb{N}$, the set*

$$\mathfrak{S}_{NM} = \left\{ \psi_z = \sum_{m=0}^{M} \sum_{j=0}^{N} Q^j(z)\phi_m : z \in J \right\} \tag{6.2}$$

is a closed subset of $\mathfrak{H}$.

Proof. The proof follows from the proof of Proposition 6.1. □

Corollary 6.5. *For any finite $n, m \in \mathbb{N}$, the set*

$$\mathfrak{S}_3 = \bigcup_{\tau=1}^{n} \bigcup_{\nu=1}^{m} \mathfrak{S}_{N_\tau M_\nu}$$
$$= \left\{ \psi_{z,\tau,\nu} = \sum_{j=0}^{N_\tau} \sum_{l=0}^{M_\nu} Q^j(z)\phi_l : z \in J, \tau = 1, 2, \ldots, n, \nu = 1, \ldots, m \right\}$$

is a closed subset of $\mathfrak{H}$.

Proposition 6.6. *For each $N \in \mathbb{N}$ let*

$$J \xrightarrow{f} \mathfrak{S}_N \xrightarrow{g} \mathbb{R}$$

by

$$f(z) = \psi_z \quad \text{and} \quad g(\psi_z) = \langle \psi_z | \psi_z \rangle = \sum_{m=0}^{N} |Q^j(z)|^2.$$

Then f is one-to-one, onto and continuous, g is continuous and

$$g \circ f(z) = \sum_{m=0}^{N} |Q^j(z)|^2$$

is continuous.

Proposition 6.7. *Let Q, J, f and $\mathfrak{S}_{MN}$ as before. Then, the following diagram*:

$$\begin{array}{ccc} J & \xrightarrow{Q} & J \\ f\downarrow & & f\downarrow \\ \mathfrak{S}_{NM} & \xrightarrow{f_Q} & \mathfrak{S}_{NM} \end{array} \tag{6.3}$$

commutes, where $f_Q(\psi_z) = \psi_{z'}$ and $z' = Q(z) \in J$.

Proof. Since

$$\psi_{Q(z)} = \psi_{z'} = \sum_{j=1}^{N}\sum_{m=0}^{M} Q^j(z')\phi_m \in \mathfrak{S}_{MN},$$

diagram (6.3) commutes. □

7. Brain imaging and wavelets

In recent years, wavelets have been constructed on 2-spheres and shown to be of central importance in many applications such as geographical data, atmospheric data, the illumination algorithms in computer graphics and medical problems of sphere-like structured organs [6,12]. A 2-sphere can be obtained by rotating a circle. The Julia set of $Q(z) = z^2 + c$, for a very small c, is a perturbed circle. If we rotate this perturbed circle we obtain a perturbed sphere which can be made arbitrarily close to any human brain by changing the value of c.

In case, one arrive to obtain a wavelet basis on these perturbed spheres, it will be very useful in the study of brain imaging. Our frame construction on Julia sets may eventually pave a way to construct such wavelet bases.

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